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Decoherence of Kicked Beams

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Introduction

When a beam is kicked transversely from the closed orbit, it begins making betatron oscillations about the closed orbit. The oscillation can be observed with beam position monitors, which give the centroid of the particles in the beam. If the particles all have the same betatron tune, the observed centroid motion is harmonic. However, if the beam contains a spread of tunes, the motion will decohere as the individual betatron phases of the particles disperse. The phase space distribution of the beam spreads from a localized bunch to an annulus which occupies all betatron phases, and the observed centroid of the beam will show a decaying oscillation.

This note will consider decoherence due to two sources of betatron tune spread: The beam bunch may have an intrinsic betatron tune spread due to transverse nonlinearity, and there may be an additional tune spread due to the energy spread of the beam which is coupled to betatron tune through the chromaticity.

Both of these problems can be solved exactly, using appropriate assumptions. In the case of transverse nonlinearity, we shall assume that the transverse distribution is Gaussian. This implicitly assumes that the distortion of phase space trajectories due to the nonlinearity is small. Also assume that the tune shift with betatron amplitude is a quadratic function.

For the case of decoherence due to chromaticity, we shall assume that the synchrotron motion is linear and that the energy distribution is Gaussian. Also assume that the energy distribution is uncorrelated with the transverse distribution, so that the chromaticity decoherence acts on each small cell of betatron phase space independently of the others. Then the decoherence due to chromaticity is completely independent of the transverse distribution.

Decoherence due to Chromaticity

The betatron tune shift of a particle due to chromaticity is

$$\Delta\nu(N) = \xi\ \delta \cos(2\pi\nu_s N + \phi_s)\ , \tag{1}$$

where ξ is the chromaticity, N is the time measured in turns, δ is the synchrotron amplitude, and ϕ_s is the synchrotron phase at $N = 0$. The synchrotron amplitude is in relative energy units, with the actual maximum energy displacement of the particle being δ times the nominal energy E_0. Let σ_s be the rms relative energy spread, so that the actual rms energy spread is $\sigma_s E_0$. Then, the betatron phase shift is

$$\Delta\phi(\delta, \phi_s, N) = 2\pi \int_0^N dN'\ \Delta\nu(N')\ = D\ \delta\ , \tag{2}$$

$$D = 2\xi\nu_s^{-1}\ \sin \pi\nu_s N\ \cos(\pi\nu_s N + \phi_s)\ . \tag{2a}$$

First find the distribution of particles in betatron phase as the beam decoheres. This is a delta function at $N = 0$, and will spread out with time. However, after a full synchrotron cycle, the distribution will return to a delta function, since the tune shifts are sinusoidal and will average to zero over a full cycle. The distribution is found by making a change of variables in the synchrotron phase space distribution, and this can be done conveniently by representing the betatron distribution as an integral over a Dirac delta function. This method gives

$$\rho(\phi, N) = \int_0^\infty d\delta \int_0^{2\pi} d\phi_s\ \rho_s(\delta)\ \delta(\phi - \Delta\phi(\delta, \phi_s, N))\ , \tag{3}$$

with the synchrotron phase space distribution

$$\rho_s(\delta) = \frac{1}{2\pi\sigma_s^2}\ \delta\ e^{-\delta^2/2\sigma_s^2}\ . \tag{4}$$

Performing the δ integral gives

$$\rho(\phi, N) = \frac{1}{2}\int_0^{2\pi} d\phi_s\ D^{-1}\rho_s(\phi/D)\ , \tag{5}$$

with D as defined in (2a). The integral over all phases ϕ_s doubly counts the peaks of the Dirac delta function, and the factor of $1/2$ compensates this. The integral in (5) is now dispatched using the change of variables

$$u = \phi\ \tan(\pi\nu_s N + \phi_s)\ , \tag{6}$$

which gives

$$\rho(\phi, N) = \frac{\phi}{\pi\alpha^2}\ e^{-\phi^2/2\alpha^2}\int_0^{\infty} du\ e^{-\phi^2 u^2/2\alpha^2}\ = \frac{1}{\sqrt{2\pi}|\alpha|}\ e^{-\phi^2/2\alpha^2}\ , \tag{7}$$

where

$$\alpha = 2\sigma_s\xi\nu_s^{-1}\ \sin\pi\nu_s N\ . \tag{7a}$$

Now, assume that the particles were initialized with betatron amplitude a. Then, the centroid of the distribution ρ has an amplitude $\bar{a}(N) = aA_s(N)$, with the decoherence factor A_s given by

$$A_s(N) = \int_{-\infty}^{\infty} d\phi\ \cos\phi\ \rho(\phi, N)\ = e^{-\alpha^2/2}\ , \tag{8}$$

with α as defined in equation (7a). The integration limits are set to infinity to include the particles which have "lapped" those with $\phi = 0$.

This chromaticity decoherence should have no effect on the measured tune, since the betatron distribution is always symmetric about the nominal betatron phase, and hence the centroid of the distribution is always at the nominal betatron phase. The decoherence function A_s is plotted in figure 1.

Decoherence due to Nonlinearity

The betatron phase space at a given point in the accelerator can be represented in pseudo-harmonic coordinates a and ϕ, where $a = \sqrt{\beta\epsilon}/\sigma_x$ and ϵ is the Courant-Snyder invariant. Note that the amplitude has been scaled to the rms beam size, so that the actual displacement is $x = \sigma_x a \ \cos(2\pi\nu N + \phi)$. ϕ is the betatron phase of a particle at $N = 0$. The transverse distribution is then

$$\rho = \frac{1}{2\pi} \ a \ e^{-a^2/2} \ . \tag{9}$$

Now consider that at $N = 0$ the beam has been kicked by an angle $\Delta x'$. This places the center of the beam at an amplitude $Z = \beta\Delta x'/\sigma_x$. The resulting distribution for the kicked beam is

$$\rho_k(a, \phi) = \frac{1}{2\pi} \ a \ e^{-(a^2+Z^2-2aZ\cos\phi)/2} \ . \tag{10}$$

Now introduce a quadratic tune dependence on amplitude:

$$\nu = \nu_0 - \mu a^2 \ . \tag{11}$$

This produces a phase slip $\Delta\phi(a, N)$ of the particle at amplitude a relative to the phase of a particle at the linear tune ν_0:

$$\Delta\phi(a, N) = -2\pi \ \mu a^2 \ N \ . \tag{12}$$

The centroid $\bar{x}(N)$ of the betatron distribution is now given by

$$\bar{x}(N) = \sigma_x \int_0^\infty da \int_0^{2\pi} d\phi \; a \;\cos\phi \; \rho_k(a, \phi - 2\pi\nu_0 N - \Delta\phi(a,N)) \; . \tag{13}$$

The ϕ integral is a representation of the modified Bessel function, giving

$$\bar{x}(N) = \sigma_x \int_0^\infty da \; a^2 \; e^{-(a^2+Z^2)/2} \; \cos(2\pi\nu_0 N + \Delta\phi(a,N)) \; I_1(aZ) \; . \tag{14}$$

This integral is then done using Gradshteyn and Ryzhik formula 6.631.4 to get

$$\bar{x}(N) = \sigma_x \; \bar{a}(N) \; \cos(2\pi\nu_0 N + \Delta\bar{\phi}(N)) \; , \tag{15}$$

where $\bar{a}(N) = Z \; A(N)$ is the amplitude of the centroid, A is the decoherence factor

$$A(N) = \frac{1}{1+\theta^2} \; \exp\left[-\frac{Z^2}{2} \frac{\theta^2}{1+\theta^2} \right] \; , \tag{15a}$$

and $\Delta\bar{\phi}$ is the phase shift of the centroid,

$$\Delta\bar{\phi}(N) = -\frac{Z^2}{2}\frac{\theta}{1+\theta^2} \; - \; 2\arctan\theta \; . \tag{15b}$$

The time dependence is contained in θ, given by

$$\theta = 4\pi \; \mu \; N \; . \tag{15c}$$

The decoherence factor proceeds as a Gaussian at short times, and then changes to a $1/N^2$ power law. Initially, $A = 1$ and the amplitude is $\bar{a}(0) = Z$.

The transition from Gaussian to power law behavior occurs at a characteristic tune (i.e. inverse number of turns)

$$Q_p = 4\pi\ \mu\ , \tag{16}$$

which gives $\theta = Q_p N$. The Gaussian part of the decoherence proceeds with a second characteristic tune

$$Q_g = Z\ Q_p\ = 4\pi\ \mu\ Z\ . \tag{17}$$

This indicates that the Gaussian decoherence is only observed for $Z \gg 1$, when the beam is kicked to an amplitude substantially greater than the beam size. Then the decoherence factor is approximately

$$A(N) \approx e^{-(Q_g N)^2/2}\ . \qquad (Z \gg 1) \tag{18}$$

In the case of small kicks, $Z \ll 1$, and the decoherence factor is approximately

$$A(N) \approx \frac{1}{1+(Q_p N)^2}\ . \qquad (Z \ll 1) \tag{19}$$

The general decoherence factor can be rewritten as

$$A(N) = \frac{1}{1+(Q_p N)^2}\ \exp\left[-\frac{1}{2}(Q_g N)^2 \frac{1}{1+(Q_p N)^2}\right]\ . \tag{20}$$

The decoherence factor A is plotted for several values of Z in figure 2.

The tune shift of the centroid motion $\Delta\bar{\nu}$ is obtained as the derivative of $\Delta\bar{\phi}$ with respect to $2\pi N$:

$$\Delta\bar{\nu}(N) = -\mu Z^2 \frac{1-(Q_p N)^2}{(1+(Q_p N)^2)^2}\ -\ 4\mu \frac{1}{1+(Q_p N)^2}\ . \tag{21}$$

The tune shift of a single particle for a kick of amplitude Z is just $-\mu Z^2$. Note that at short times after the kick, i.e. for $Q_p N \ll 1$, The centroid tune shift contains an offset: $\Delta\bar{\nu} \approx -\mu Z^2 - 4\mu$. The offset -4μ is equivalent to the tune shift of a particle at an amplitude of twice the rms beam size, and for small kicks this offset dominates the observed tune shift. For $Q_p N \gg 1$, $\Delta\bar{\nu} \to 0$. This is because the decoherence proceeds most slowly in the region of phase space near the origin, where the slope of the tune shift with amplitude is smallest. Hence the structure of the kicked beam persists longest near the origin, and the particles in this region have tune shifts near zero.

Combined Decoherence

The consequence of both of the above effects acting together can be easily seen. Since the chromaticity and nonlinearity effects are independent, then the actual distribution in betatron phase ϕ is the convolution of the two distributions (7) and (10). The computation of the centroid displacement $\bar{x}$ in equation (13) has the mathematical form of a Fourier transform, and since the distribution ρ_k is now a convolution, the resulting decoherence factor is just the product of the decoherence factors A_s in equation (8) and A in equation (20).

Measurement of Tune

Consider the problem of measuring the tune of the accelerator by kicking a bunch to amplitude Z and observing the centroid signal $\bar{x}(N)$. This technique might be used to determine the constant μ, for example. For a kick of Z, the tune shift of a single particle would be $\Delta\nu = -\mu Z^2$. However, the centroid tune of a many particle bunch $\Delta\bar{\nu}$ is a function of time as shown in (21). Two methods of obtaining a measured tune $\Delta\nu_m$ are by locating the peak in a Fourier transform of $\bar{x}(N)$, and by counting cycles of $\bar{x}(N)$.

Consider first the peak of the Fourier transform. The location of this peak can be found by examining the Fourier transform of equation (14). Only the real part of the fourier transform can be found easily, but this is sufficient since the imaginary part passes through zero near the peak. This is because the function

$\bar{x}(N) = 0$ for $N < 0$, and so the real and imaginary parts of the Fourier transform are connected by the Hilbert transformation, which associates zero crossings with peaks. Let $X(\nu)$ be the transformed beam signal. Then

$$\mathrm{Re}\; X(\nu) = \sigma_x \int\limits_0^\infty da\; a^2\; e^{-(a^2+Z^2)/2}\; \delta(\nu - \nu_0 + \mu a^2)\; I_1(aZ)\;. \tag{22}$$

This trivial integral gives

$$\mathrm{Re}\; X(\nu) \begin{cases} = C\; a\; e^{-(a^2+Z^2)/2}\; I_1(aZ)\;, & \text{for } |\nu| < \nu_0; \\ = 0\;, & \text{for } |\nu| \geq \nu_0, \end{cases} \tag{23}$$

where C is an irrelevant constant, and the tune ν is implicitly represented by the equivalent amplitude a:

$$\nu = \nu_0 \;-\; \mu a^2\;. \tag{23a}$$

Differentiating in ν and setting to zero to obtain the peak gives the equation for the measured tune shift $\Delta\nu_m$, again represented implicitly by the equivalent amplitude a_m:

$$Z^2 = a_m Z\; I_1(a_m Z)/I_0(a_m Z)\;. \tag{24}$$

A plot of a_m versus Z is given in figure 3. The asymptotic form of the solution for large Z is

$$a_m \approx Z + \frac{1}{2Z}\;, \qquad (Z \gg 1) \tag{25}$$

and the small kick result is

$$a_m \approx \sqrt{2}\;. \qquad (Z \ll 1) \tag{26}$$

In the case that the tune is measured by counting cycles of $\bar{x}(N)$, the measured tune is the phase shift at the measurement time divided by $2\pi N$:

$$\Delta\nu_m = -\mu Z^2 \frac{1}{1 + (Q_p N_m)^2} \;-\; 4\mu\; (Q_p N_m)^{-1} \arctan(Q_p N_m)\;, \tag{27}$$

where N_m represents the turn of measurement. This measured tune has the same qualitative properties as the centroid tune $\Delta\bar{\nu}$: For small N,

$$\Delta\nu_m \approx -\mu Z^2 \; - \; 4\mu \; , \qquad (Q_p N \ll 1) \quad (28)$$

and for large N,

$$\Delta\nu_m \to 0 \; . \qquad (Q_p N \gg 1) \quad (29)$$

The measured tune, represented as equivalent amplitude a_m, is plotted in figures 4–6. Figure 4 is a plot of a_m versus Z for fixed measurement time N_m, where the dotted lines indicate that $A < e^{-2}$. Figure 5 is a plot of a_m versus Z for fixed decoherence factor A, and figure 6 is a plot of a_m versus decoherence factor A for fixed Z.

Conclusions

The centroid of a kicked beam can be calculated as the beam decoheres with time. The required assumptions are that the beam is Gaussian in width and energy, that the chromaticity is linear, and that the betatron tune is quadratic in amplitude. Let the relative energy width be σ_s, the tune shift with amplitude be $-\mu\; a^2$, and the kick amplitude be Z, where both a and Z are normalized to the rms beam size. Then the resulting centroid motion is

$$\bar{x}(N) = \sqrt{\beta\epsilon}\; A_s(N)\; A(N)\; \cos(2\pi\nu_0 N + \Delta\bar{\phi}(N)) \; , \qquad (30)$$

where A_s and A are decoherence factors due to chromaticity and nonlinearity, respectively:

$$A_s(N) = e^{-\alpha^2/2} \; , \qquad (31)$$

$$\alpha = 2\sigma_s \xi \nu_s^{-1} \; \sin \pi\nu_s N \; , \qquad (31a)$$

$$A(N) = \frac{1}{1+(Q_p N)^2} \; \exp\left[-\frac{1}{2}(Q_g N)^2 \frac{1}{1+(Q_p N)^2}\right] \; , \qquad (32)$$

where Q_p and Q_g are characteristic tunes (or inverse turn numbers) which mark the progress of the decoherence:

$$Q_p = 4\pi\ \mu\ , \quad Q_g = Z\ Q_p\ . \tag{33}$$

The centroid phase shift is

$$\Delta\bar{\phi}(N) = -\frac{Z^2}{2}\frac{Q_pN}{1+(Q_pN)^2}\ -\ 2\arctan(Q_pN)\ . \tag{34}$$

From these formulae, the results of amplitude and frequency measurements of a kicked beam can be predicted.

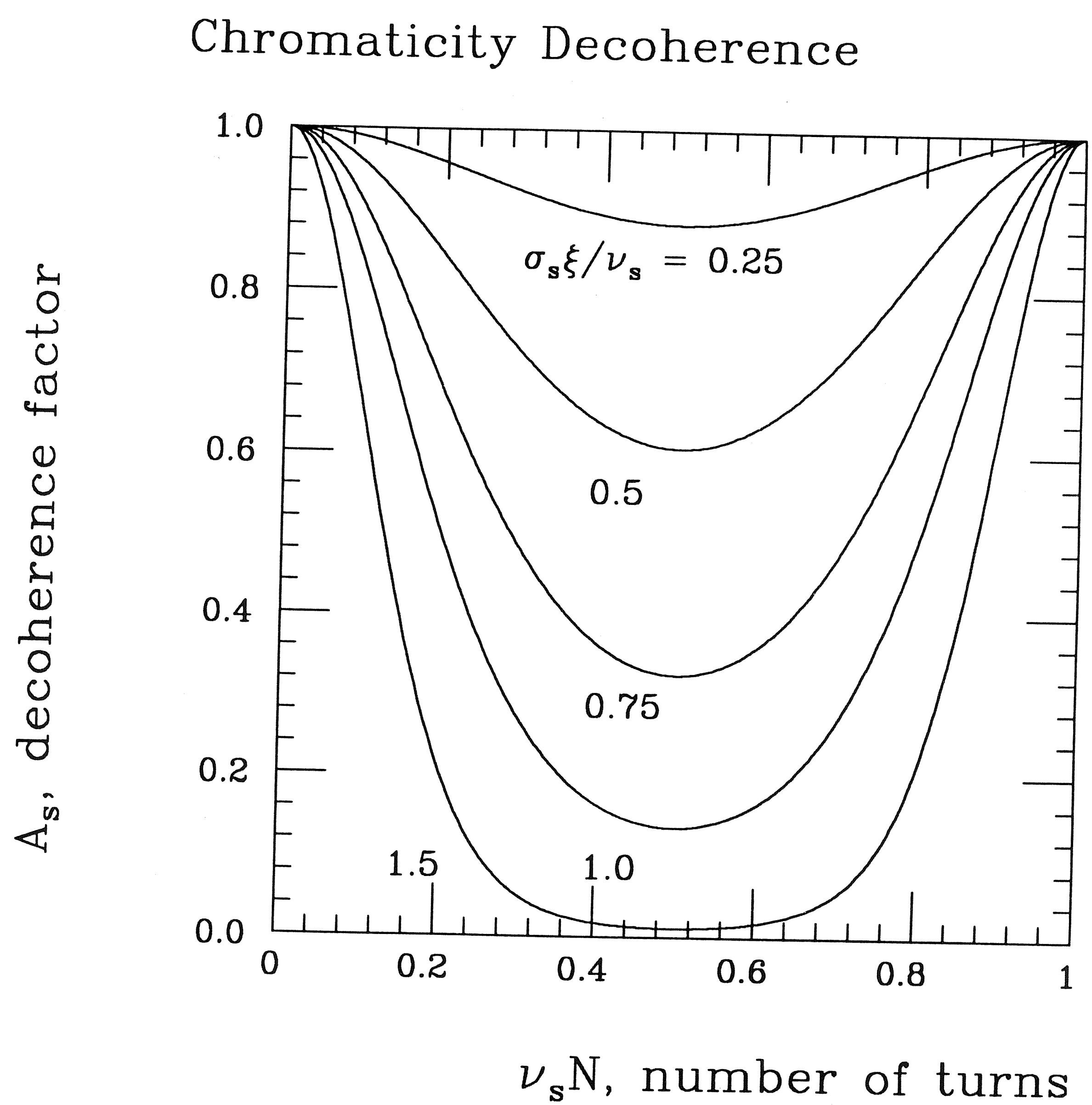
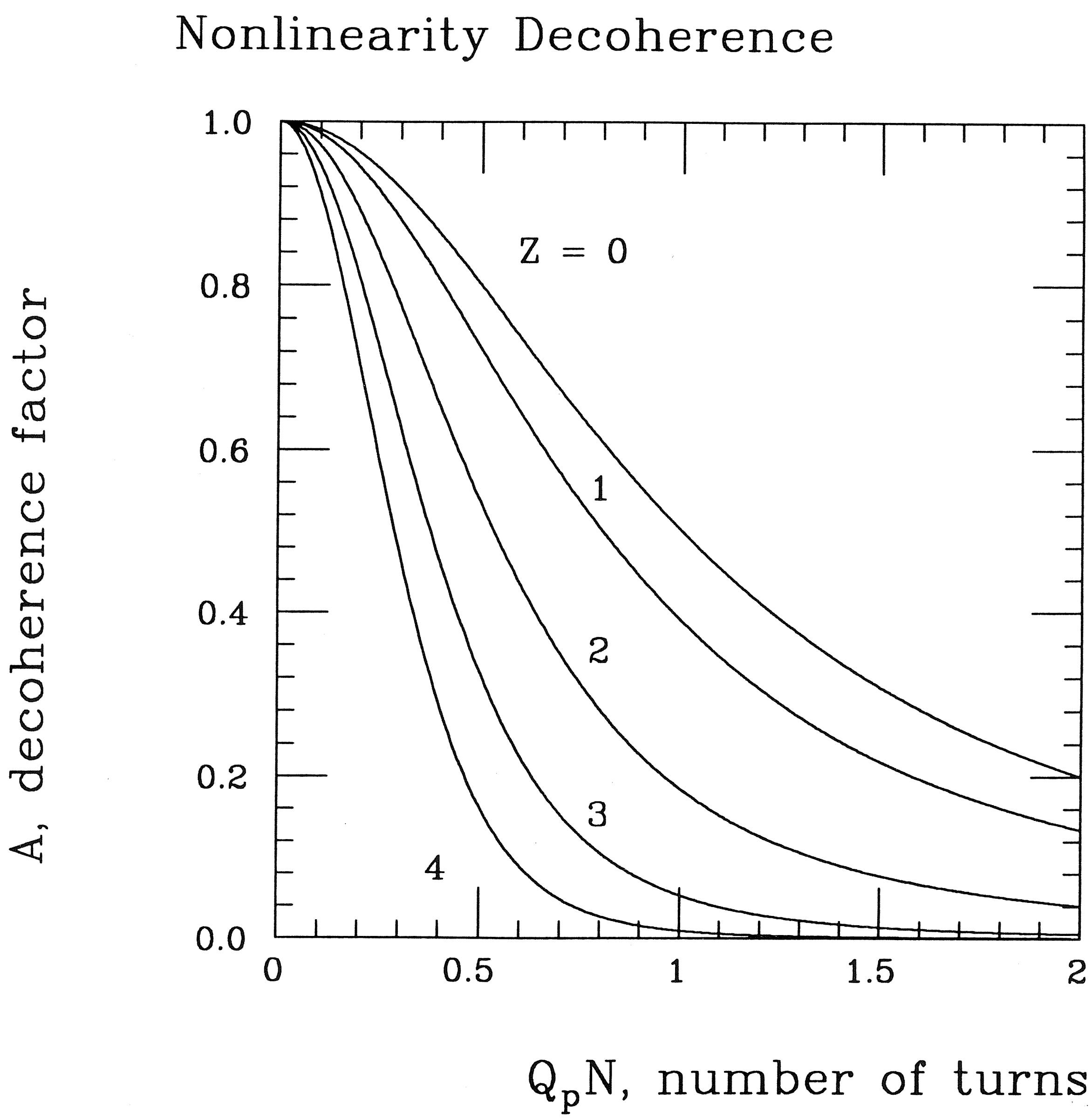

Chromaticity Decoherence
$\sigma_s\xi/\nu_s$ = 0.25
0.5
0.75
1.0
1.5
1.0
0.8
0.6
0.4
0.2
0.0
0
0.2
0.4
0.6
0.8
1
A_s, decoherence factor
ν_sN, number of turns

Figure 1



Nonlinearity Decoherence
1.0
0.8
0.6
0.4
0.2
0.0
0
0.5
1
1.5
2
Z = 0
1
2
3
4
A, decoherence factor
$Q_p N$, number of turns

Figure 2

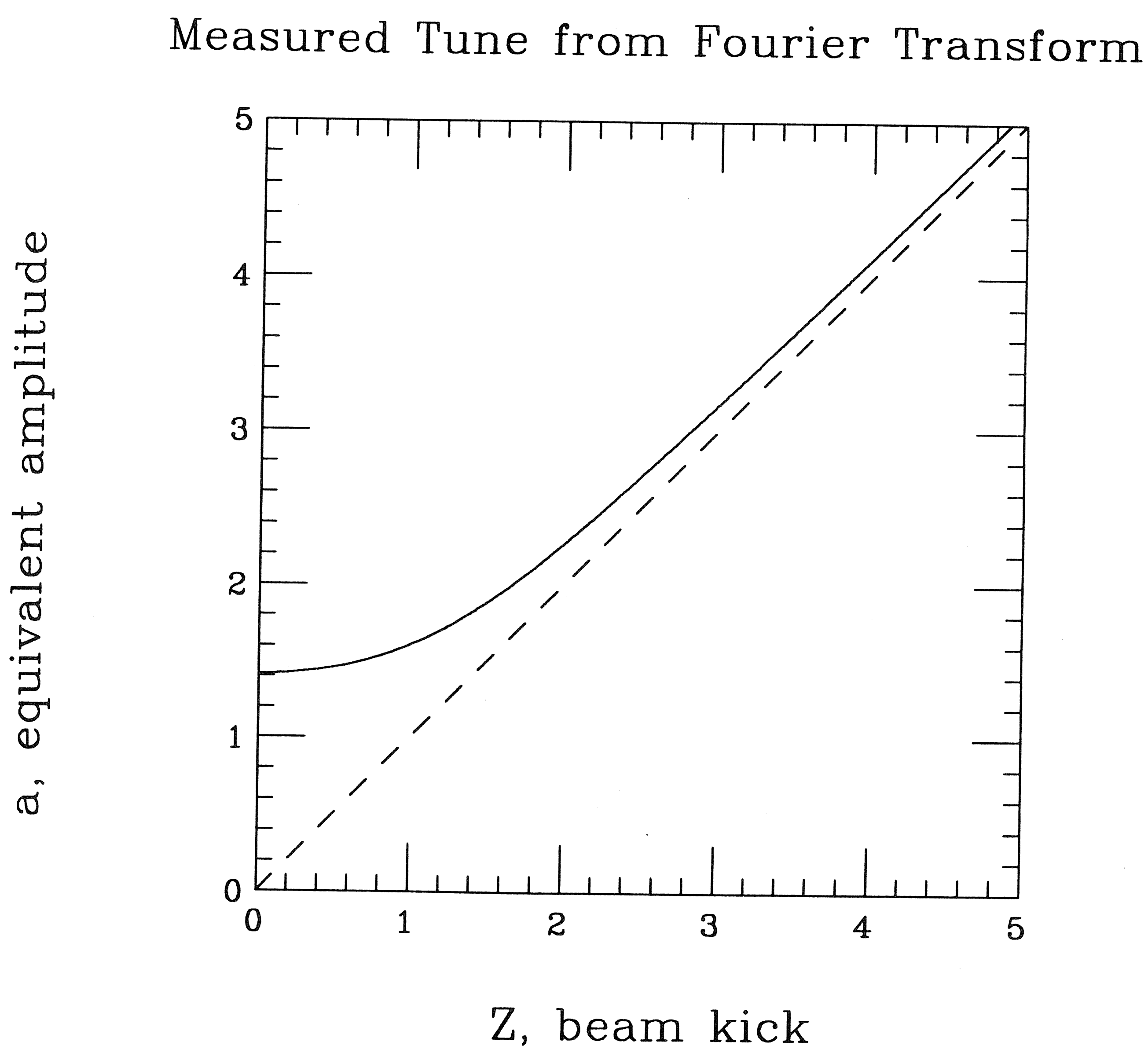

Measured Tune from Fourier Transform
a, equivalent amplitude
Z, beam kick
0
1
2
3
4
5

Figure 3

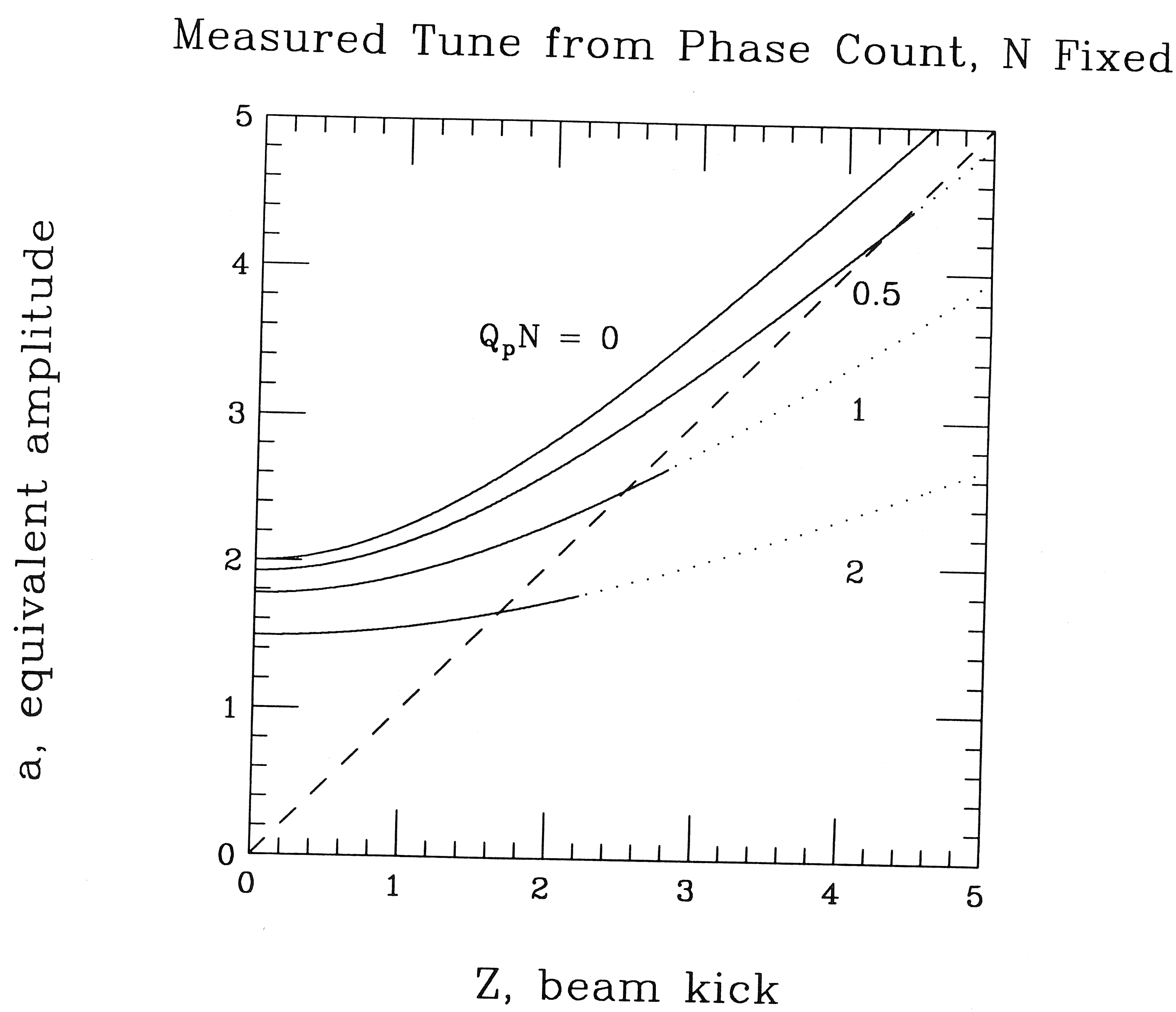

Measured Tune from Phase Count, N Fixed
a, equivalent amplitude
Z, beam kick
$Q_pN = 0$
0.5
1
2
0
1
2
3
4
5

Figure 4

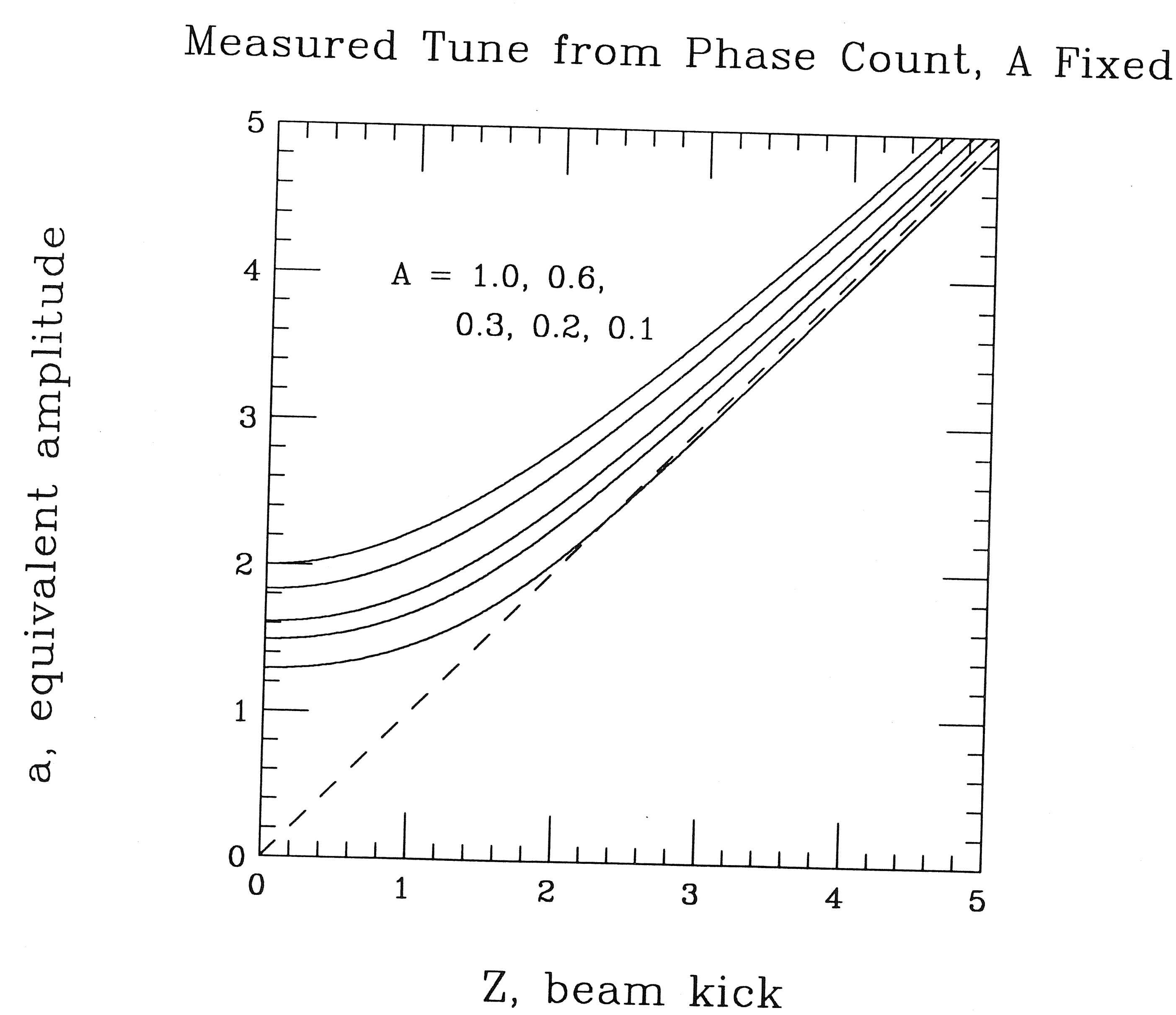

Measured Tune from Phase Count, A Fixed
A = 1.0, 0.6,
0.3, 0.2, 0.1
a, equivalent amplitude
Z, beam kick
0
1
2
3
4
5

Figure 5

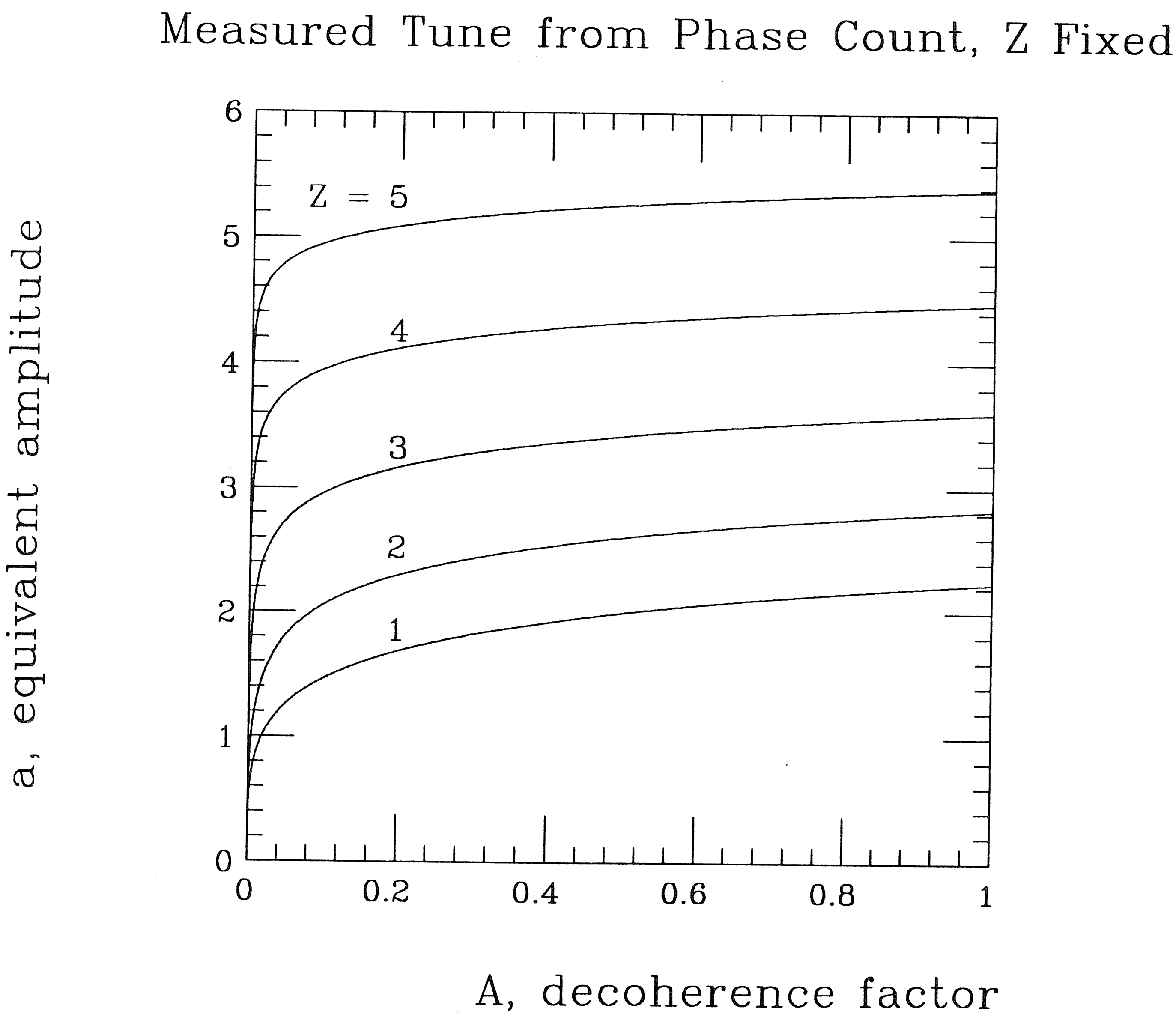

Measured Tune from Phase Count, Z Fixed
a, equivalent amplitude
A, decoherence factor
Z = 5
4
3
2
1
0
1
2
3
4
5
6
0
0.2
0.4
0.6
0.8
1

Figure 6